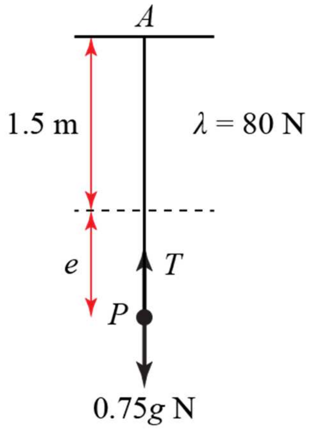
Exercise 3E

1 a



In equilibrium:

$R(\uparrow)\ T = 0.75g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

Resolve and use Hooke's Law with the equilibrium extension (e).

$$T = \frac{80e}{1.5}$$

$$\therefore 0.75g = \frac{80e}{1.5}$$

$$e = 0.75 \times 9.8 \times \frac{1.5}{80}$$

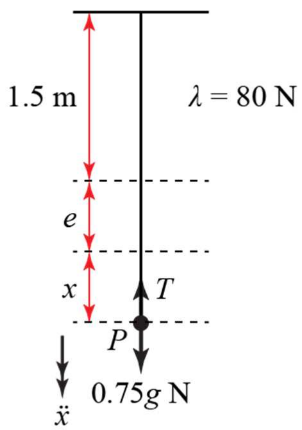
$$= 0.1378\ldots$$

$$e + l = 1.637\ldots$$

The total length of the spring is required.

The length of the spring is 1.64 m (3 s.f.)

b



$$F = ma$$

$$0.75g - T = 0.75\ddot{x}$$

x is measured from the equilibrium level, and $\ddot{x}$ is in the direction of increasing x.

Hooke's Law:

$$T = \frac{80(x+e)}{1.5}$$

$$\therefore 0.75g - \frac{80(x+e)}{1.5} = 0.75\ddot{x}$$

To avoid decimals use $(e + x)$ for the extension where $\dfrac{80e}{1.5} = 0.75g$ (from **a**)

from **a** $0.75g = \dfrac{80\text{ e}}{1.5}$

$$\therefore 0.75\ddot{x} = -\frac{80}{1.5}x$$

$$\ddot{x} = -\frac{80}{1.5 \times 0.75}x$$

of the form $\ddot{x} = -\omega^2 x$

$\therefore$ S.H.M.

c

$$\omega^2 = \frac{80}{1.5 \times 0.75}$$

$$\text{Period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1.5 \times 0.75}{80}}$$

$$= 0.7450\ldots$$

The period is 0.745s (3 s.f.)

1 d $v^2 = \omega^2(a^2 - x^2)$

$x = 0$ at the equilibrium level.

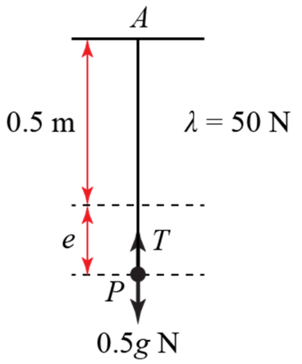
$$2.5^2 = \frac{80}{1.5 \times 0.75}a^2$$

$$a^2 = \frac{2.5^2 \times 1.5 \times 0.75}{80}$$

$$a = 0.2964...$$

The amplitude is 0.296 m (3 s.f.)

2 a



$R(\uparrow)\ T = 0.5g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{50e}{0.5}$$

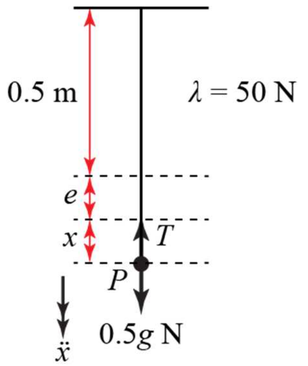
$$T = 100e$$

$$\therefore 100e = 0.5g$$

$$e = \frac{0.5 \times 9.8}{100} = 0.049$$

The extension is 0.049 m (or 4.9 cm)

b



$$F = ma$$

$$0.5g - T = 0.5\ddot{x}$$

Use $F = ma$ and Hooke's Law to find ω.

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{50(e + x)}{0.5}$$

$$T = 100(e + x)$$

$$\therefore 0.5g - 100(e + x) = 0.5\ddot{x}$$

from **a** $100e = 0.5g$

$$\therefore \quad -100x = 0.5\ddot{x}$$

$$\ddot{x} = -200x$$

$$\omega^2 = 200$$

Compare previous line with $\ddot{x} = -\omega^2 x$.

$$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{200}} = \frac{2\pi}{10\sqrt{2}} = \frac{\pi}{10}\sqrt{2}$$

The period is $\dfrac{\pi}{10}\sqrt{2}$ s (or 0.444s (3 s.f.)).

c amplitude = 0.2 m

$$v^2 = \omega^2(a^2 - x^2)$$

$$v_{max} = \omega a$$

The maximum speed occurs at the equilibrium level (i.e. when $x = 0$).

$$= \sqrt{200} \times 0.2$$

$$= 2\sqrt{2}$$

The maximum speed is $2\sqrt{2}$ m s^{-1} (or 2.83 m s^{-1} (3 s.f.)).

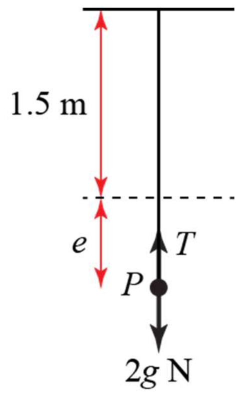
3 **a** For the impact: $I = mv - mu$

$$3 = 2v$$

$$v = 1.5$$

The speed immediately after the impact is 1.5 m s^{-1}.

b



In equilibrium

$R(\uparrow)\ T = 2g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{\lambda e}{1.5}$$

$$\frac{\lambda e}{1.5} = 2g$$

λ is unknown so will remain in this expression.

When oscillating:

$$F = ma$$

$$2g - T = 2\ddot{x}$$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$$T = \frac{\lambda(e+x)}{1.5}$$

$$\therefore\ 2g - \frac{\lambda(e+x)}{1.5} = 2\ddot{x}$$

From above: $\dfrac{\lambda e}{1.5} = 2g$

$$\therefore\ -\frac{\lambda x}{1.5} = 2\ddot{x}$$

$$\ddot{x} = -\frac{\lambda}{3}x$$

as $\lambda > 0$, this is S.H.M.

c period $= \dfrac{2\pi}{\omega} = \dfrac{\pi}{2}$

$\therefore\ \omega = 4$

From $\ddot{x} = -\dfrac{\lambda}{3}x$, $\omega^2 = \dfrac{\lambda}{3}$

$$\therefore\ \frac{\lambda}{3} = 16$$

$$\lambda = 48$$

3 d maximum speed = 1.5 m s^{-1}

$v^2 = \omega^2(a^2 - x^2)$

$v_{max} = \omega a$

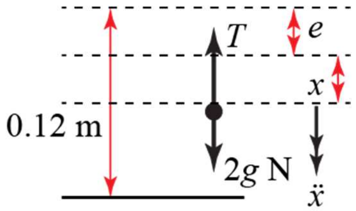
Maximum speed occurs when $x = 0$.

$1.5 = 4a$

$a = \frac{1.5}{4} = 0.375$

The amplitude is 0.375 m.

4 a



$\lambda = 500\text{ N}$

In equilibrium:

$R(\uparrow)\; T = 2g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{500e}{l}$

$\therefore 2g = \frac{500e}{0.12}$

Change cm to m.

For the oscillations:

$F = ma$

$2g - T = 2\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$T = \frac{500(e+x)}{0.12}$

$\therefore\; 2g - \frac{500(e+x)}{0.12} = 2\ddot{x}$

From above: $\frac{500e}{0.12} = 2g$

$\therefore\; -\frac{500x}{0.12} = 2\ddot{x}$

$\ddot{x} = -\frac{250}{0.12}x$

$\omega^2 = \frac{250}{0.12}$

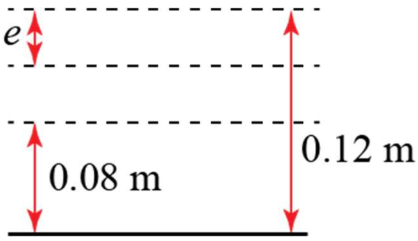
Compare line above with $\ddot{x} = -\omega^2 x$.

period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{0.12}{250}}$

$= 0.1376\ldots$

The period is 0.138 s (3 s.f.)

4 b $e = \dfrac{2g \times 0.12}{500}$



amplitude = $0.04 - e$

$v^2 = \omega^2(a^2 - x^2)$

$v_{\text{max}} = a\omega$ ← Maximum speed occurs when $x = 0$.

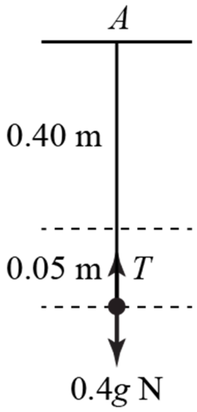
$= \sqrt{\dfrac{250}{0.12}} \times (0.04 - e)$

$= \sqrt{\dfrac{250}{0.12}} \times \left(0.04 - \dfrac{2g \times 0.12}{500}\right)$

$= 1.611\ldots$

The maximum speed is 1.61 m s^{-1} (3 s.f.).

5 a



In equilibrium:

$R(\uparrow) T = 0.4g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{\lambda 0.05}{0.4}$

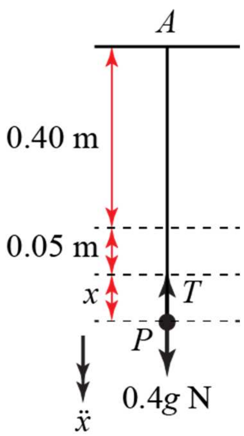
$\therefore \lambda \times \dfrac{0.05}{0.4} = 0.4g$

$\lambda = \dfrac{0.4^2}{0.05} g$

$\lambda = 3.2g = 31.36$

The modulus of elasticity is 31.4 N (3 s.f.)

b



For oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{31.36(x + 0.05)}{0.4}$

$0.4g - \dfrac{31.36(0.05)}{0.4} = 0.4\ddot{x}$

$\ddot{x} = -\dfrac{31.36}{0.4^2} x$

$\therefore$ S.H.M.

5 c From $\ddot{x} = -\frac{31.36}{0.4^2}x$

$$\omega = \frac{\sqrt{31.36}}{0.4}$$

$$\text{period} = \frac{2\pi}{\omega} = 2\pi \times \frac{0.4}{\sqrt{31.36}} = 0.4487\ldots$$

The period is 0.449 s.

amplitude = 52 − 45 = 7 (cm)

The amplitude is 0.07 m.

d $v^2 = \omega^2(a^2 - x^2)$

$$v_{\max} = \omega a$$
$$= \frac{\sqrt{31.36}}{0.4} \times 0.07$$
$$= 0.98$$

The maximum speed is 0.98 m s^{-1}.

e 11 cm from the lowest point

$$\Rightarrow AP = 41\,\text{cm}.$$

$$\therefore\ x = -4 \text{ cm} = -0.04 \text{ m}$$

$$x = a\cos\omega t$$

P starts from an end point.

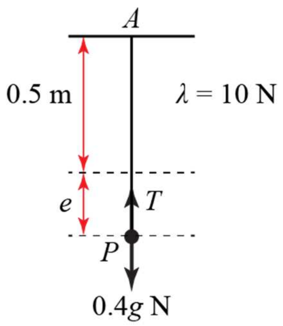
$$-0.04 = 0.07\cos\omega t$$

$$\omega t = \cos^{-1}\left(-\frac{0.04}{0.07}\right) = \cos^{-1}\left(-\frac{4}{7}\right)$$

$$t = \frac{1}{\omega}\cos^{-1}\left(-\frac{4}{7}\right) = \frac{0.4}{\sqrt{31.36}}\cos^{-1}\left(-\frac{4}{7}\right)$$
$$= 0.1556\ldots$$

P takes 0.156 s to rise 11 cm (3 s.f.).

6 a



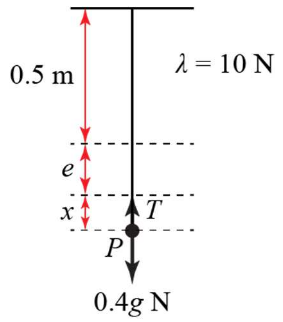
In equilibrium:

$R(\uparrow) T = 0.4g$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{10e}{0.5} = 20e$

$\therefore 20e = 0.4g$



For the oscillations:

$F = ma$

$0.4g - T = 0.4\ddot{x}$

Hooke's Law: $T = \dfrac{\lambda x}{l}$

$T = \dfrac{10(e+x)}{0.5}$

$\therefore 0.4g - \dfrac{10(e+x)}{0.5} = 0.4\ddot{x}$

From above $0.4g = \dfrac{10e}{0.5}$

$\therefore \quad -\dfrac{10x}{0.5} = 0.4\ddot{x}$

$\ddot{x} = -\dfrac{20x}{0.4} = -50x$

$\therefore$ S.H.M. with $\omega^2 = 50$

amplitude $= 0.2\,\text{m}$

$x = a\cos\omega t$

$x = 0.2\cos\sqrt{50}t$

String becomes slack when $x = -e$

$-\dfrac{0.4g}{20} = 0.2\cos\sqrt{50}t$

$\cos\sqrt{50}t = -\dfrac{2g}{20} = -\dfrac{g}{10} = -0.98$

$\sqrt{50}t = \cos^{-1}(-0.98)$

$t = \dfrac{1}{\sqrt{50}}\cos^{-1}(-0.98)$

$t = 0.4159$

The string becomes slack after 0.416 s (3 s.f.)

6 b The velocity of P is given by $v^2 = \omega^2\left(a^2 - x^2\right)$

where $\omega^2 = 50$, $a = 0.2$ and $x = 0.196$

Therefore at the instant the string becomes slack

$v^2 = 50\left(0.2^2 - 0.196^2\right)$

$v = 0.2814\ldots \text{ m s}^{-1}$

At the instant the string becomes slack, the particle is no longer moving under SHM but under gravity. To find the time taken for P to pass through this point again use

$v = u + at$

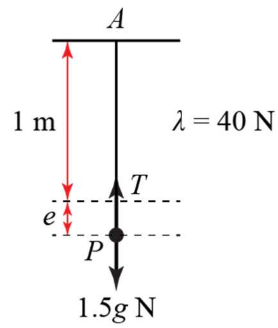
with $v = -0.2814$, $u = 0.2814$ and $a = -9.8$

$-0.2814 = 0.2814 - 9.8t$

$t = 0.5742\ldots$ s

So the time taken for the string to become taut again is 0.574 s (3 s.f.)

7 a



In equilibrium:

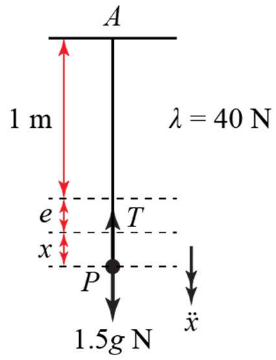
$R(\uparrow)\ T = 1.5g$

Hooke's Law: $T = \frac{\lambda x}{l}$

$1.5g = \frac{40e}{1}$

$e = \frac{1.5e}{40} = 0.3675\,\text{m}$

a can be done by using conservation of energy but **b** needs S.H.M. So S.H.M. has been used for both parts.



For the oscillation:

$F = ma$

$1.5g - T = 1.5\ddot{x}$

Hooke's Law: $T = \frac{\lambda x}{l}$

$= \frac{40(x+e)}{1}$

$\therefore 1.5g - 40(x+e) = 1.5\ddot{x}$

From above $1.5g = 40e$

$\therefore \quad 1.5\ddot{x} = -40x$

$\ddot{x} = -\frac{80}{3}x$

$\omega = \sqrt{\frac{80}{3}}$

amplitude $= 0.8 - 0.3675 = 0.4325\,\text{m}$

$v^2 = \omega^2(a^2 - x^2)$

When the string is cut: $x = 0.4325 - 0.4$

$= 0.0325$

and $v^2 = \frac{80}{3}(0.4325^2 - 0.0325^2)$

$= 4.96$

Find the speed when the string is cut.

Motion under gravity:

$v^2 = u^2 + 2as$

$0 = 4.96 - 2 \times 9.8s$

$s = \frac{4.96}{2 \times 9.8} = 0.2530\ldots$

Use motion under gravity.

height above equilibrium position

$= 0.2530 - 0.0325 = 0.2205$

Height is 0.221 m.

7 b For S.H.M. $x = a\cos\omega t$

Particle starts from an end-point.

$$x = 0.4325\cos\sqrt{\frac{80}{3}}t$$

$$x = 0.0325 \quad 0.0325 = 0.4325\cos\sqrt{\frac{80}{3}}t$$

$$\cos\sqrt{\frac{80}{3}}t = \frac{0.0325}{0.4325}$$

$$t = \sqrt{\frac{3}{80}}\cos^{-1}\left(\frac{0.0325}{0.4325}\right)$$

$$= 0.2896$$

Motion under gravity:

$$v = u + at$$

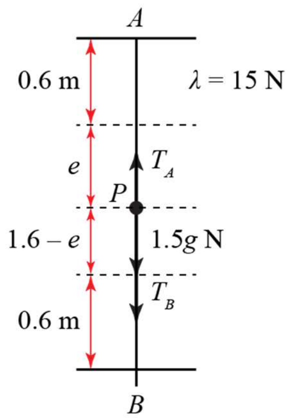
$$O = \sqrt{4.96} - 9.8t$$

$$t = \frac{\sqrt{4.96}}{9.8}$$

$$\text{total time } = 0.2896\ldots + \frac{\sqrt{4.96}}{9.8} = 0.5168\ldots$$

The time taken to reach the highest point is 0.517s (3 s.f.)

8 a



In equilibrium

$$R(\uparrow) T_A = 1.5g + T_B$$

Hooke's Law:

$$T = \frac{\lambda x}{l}$$

$$T_A = \frac{15e}{0.6} = 25e$$

$$T_B = \frac{15(1.6 - e)}{0.6} = 40 - 25e$$

You must consider P to be attached to two strings. The tensions in the two parts will be different.

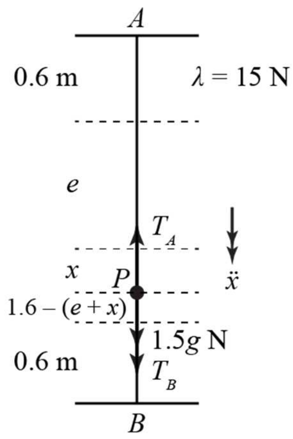
$$\therefore\ 25e = 1.5g + 40 - 25e$$

$$50e = 1.5g + 40$$

$$e = \frac{1}{50}(1.5g + 40) = 1.094$$

In equilibrium, $AP = 1.69\,\text{m}$ (3s.f.)

8 b



For the oscillations:

$$T_A = \frac{15(e+x)}{0.6}$$

$$T_B = \frac{15(1.6-(e+x))}{0.6}$$

$$F = ma$$

$$1.5g + \frac{15(1.6-(e+x))}{0.6} - \frac{15(e+x)}{0.6} = 1.5\ddot{x}$$

$$1.5g + 40 - 25(e+x) - 25(e+x) = 1.5\ddot{x}$$

$$1.5g + 40 - 50e - 50x = 1.5\ddot{x}$$

from **a** $50e = 1.5g + 40$

$$\therefore 1.5\ddot{x} = -50x$$

$$\ddot{x} = -\frac{50}{1.5}x = -\frac{100}{3}x$$

$\therefore$ P moves with S.H.M.

c amplitude = 0.15 m

$$x = a\cos\omega t = 0.15\cos\left(\frac{10}{\sqrt{3}}\right)t$$

When $x = -0.1$

The equilibrium position is the centre of the oscillation.

$$-0.1 = 0.15\cos\left(\frac{10}{\sqrt{3}}T\right)$$

$$\cos\left(\frac{10}{\sqrt{3}}T\right) = -\frac{0.1}{0.15}$$

$$T = \frac{\sqrt{3}}{10}\cos^{-1}\left(-\frac{0.1}{0.15}\right)$$

$$= 0.3984\ldots$$

$\therefore$ $T = 0.398$ (3 s.f.)

9 a Until rope is taut:

Climber falling freely under gravity.

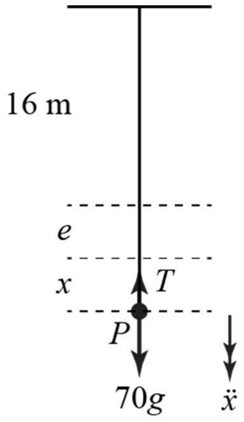
$$v^2 = u^2 + 2as$$

$$v^2 = 0 + 2\times 9.8\times 8$$

$$v = 12.52\ldots$$

When the rope becomes taut the climber's speed is 12.5 m s^{-1} (3 s.f.)

9 b



At the equilibrium level:

Hooke's Law: $T = \dfrac{\lambda x}{l}$

b can be solved by using conservation of energy. However c involves time, so S.H.M. methods are needed. It is more efficient to use S.H.M. for both parts.

$$T = \frac{40\,000e}{16}$$

$$R(\uparrow)\, T = 70g$$

$$\therefore \quad 70g = \frac{40\,000e}{16}$$

$$e = \frac{16 \times 70g}{40\,000} = \frac{7g}{250}$$

For the oscillation:

$$F = ma$$

$$70g - T = 70\ddot{x}$$

Hooke's Law: $T = \dfrac{40\,000(x+e)}{16}$

$$70g - \frac{40\,000(x+e)}{16} = 70\ddot{x}$$

From **a**: $70g = \dfrac{40\,000e}{16}$

$$\ddot{x} = -\frac{4000}{16 \times 7}x = -\frac{250}{7}x$$

$$\omega^2 = \frac{250}{7}$$

$$v^2 = \omega^2(a^2 - x^2)$$

Use the result from part **a**, ie the speed when $x = e\left(= \dfrac{7g}{250}\right)$.

$$156.8 = \frac{256}{7}\left(a^2 - \left(\frac{7g}{250}\right)^2\right)$$

$$a^2 = \frac{156.8 \times 7}{250} + \left(\frac{7g}{250}\right)^2$$

$$a^2 = 4.4656\ldots$$

$$a = 2.113\ldots$$

Total distance $= 2.113 + e + 8$

$$= 2.113 + \frac{7g}{250} + 8$$

$$= 10.38\ldots$$

The amplitude is the greatest distance below the equilibrium level.

The total distance fallen is 10.4 m (3 s.f.)

9 c $x = a\cos\omega t$

$x = 2.113\cos\sqrt{\frac{250}{7}}t$

Because of the symmetry of S.H.M. there are several methods available for **c**.

When $x = e$

$\frac{7g}{250} = 2.113\cos\sqrt{\frac{250}{7}}t$

$t = \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

This method assumes the oscillation is complete and finds the time from the highest point ($x = a$) to the equilibrium level ($x = e$). This time will be subtracted from half the period. So it does not matter that this part of the oscillation does not exist.

$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{7}{250}}$

Time while the rope is taut:

$= \frac{2\pi}{2}\sqrt{\frac{7}{250}} - \sqrt{\frac{7}{250}}\cos^{-1}\left(\frac{7\times 9.8}{250\times 2.113}\right)$

$= 0.2846\ldots$

Time from highest point to lowest point of a complete oscillation is half the period. Subtract the time for the missing part (before the rope is taut) to obtain the time while the rope is taut.

While moving under gravity:

$s = ut + \frac{1}{2}at^2$

$8 = \frac{1}{2}\times 9.8t^2$

$t^2 = \frac{16}{9.8}$

The time before the rope becomes taut is also needed.

$\text{total time} = \frac{4}{\sqrt{9.8}} + 0.2846\ldots$

$= 1.562\ldots$

The total time is 1.56 s (3 s.f).

Challenge

Particle P moves with SHM, time period

$$T = 2\pi\sqrt{\frac{ml}{\lambda}} = 2\pi\sqrt{\frac{ml}{5mg}}$$

$$= 2\pi\sqrt{\frac{l}{5g}}$$

Particles P and Q together move with SHM,

time period $3T = 6\pi\sqrt{\frac{l}{5g}}$

Also $3T = 2\pi\sqrt{\frac{(m+km)l}{5mg}}$

$$= 2\pi\sqrt{\frac{l(1+k)}{5g}}$$

So $6\pi\sqrt{\frac{l}{5g}} = 2\pi\sqrt{\frac{l(1+k)}{5g}}$

$$3\sqrt{l} = 2\sqrt{l(1+k)}$$

$$9l = 4l(1+k)$$

$$9 = 4 + 4k$$

$$k = \frac{5}{4}$$